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Amplitude Death in an Array of Limit-Cycle Oscillators

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We analyze a large system of limit-cycle oscillators with mean-field coupling and randomly distributed natural frequencies. We prove that when the coupling is sufficiently strong and the distribution of frequencies has sufficiently large variance, the system undergoes "amplitude death"—the oscillators pull each other off their limit cycles and into the origin, which in this case is a *stable* equilibrium point for the coupled system. We determine the region in coupling-variance space for which amplitude death is stable, and present the first proof that the infinite system provides an accurate picture of amplitude death in the large but finite system.

KEY WORDS: Nonlinear oscillator; bifurcation; phase transition; mean-field model; self-synchronization; collective phenomena.

1. INTRODUCTION

Arrays of coupled nonlinear oscillators arise in many branches of science and technology. Recent applications in physics include phase-locked arrays of lasers,[1] Josephson junctions,[2] and relativistic magnetrons.[3] Networks of nonlinear oscillators have also been used to model the generation of biological rhythms in the heart,[4–8] nervous system,[4–7,9–11] intestine,[5,12] and pancreas.[13]

Oscillator arrays also pose extremely interesting theoretical problems that lie at the intersection of statistical mechanics and nonlinear dynamics.

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For example, consider an array of coupled limit-cycle oscillators with randomly distributed natural frequencies.[14–17] (This provides a reasonable model for the network of pacemaker cells in the heart.[4,5,8]) As the coupling is increased beyond a certain threshold, the oscillators suddenly begin to entrain one another to a common frequency. The sudden onset of self-synchronization is strikingly similar to a second-order phase transition for an equilibrium system.[4,14] In this example, the order parameter measures the temporal coherence among the oscillators; it vanishes if the coupling is too weak, and then grows sharply (but continuously) once the coupling exceeds threshold.

Many of the theoretical studies of oscillator arrays have been limited to the case of phase-only oscillators.[4,10–12,14–17] That is, the amplitude degrees of freedom of the oscillators have been neglected. This is a reasonable approximation if the system is composed of limit-cycle oscillators that are weakly coupled, relative to the attractiveness of their limit cycles.[12]

However, when amplitude variations are not negligible, a number of intriguing new phenomena can occur.[18–29] One of these is "*amplitude death*," discovered by Yamaguchi and Shimizu[19] and studied further by Ermentrout and his collaborators.[22–25] Amplitude death is a coupling-induced stabilization of the origin: the oscillators pull each other off their limit cycles, and collapse into a state of zero amplitude. Yamaguchi and Shimizu[19] found that two ingredients are needed for amplitude death to be stable: (i) sufficiently strong coupling between the oscillators and (ii) a sufficiently wide distribution of natural frequencies.

The phenomenon of amplitude death may be relevant to certain pathologies of biological oscillator networks. It corresponds to the cessation of rhythmicity in a system which is spontaneously rhythmic for other choices of parameters. Other possible mechanisms for arrhythmias are discussed in refs. 5, 6, and 30.

Since the work of Yamaguchi and Shimizu,[19] amplitude death has been studied by a number of authors.[20–27] The case of $n = 2$ oscillators has been analyzed rigorously.[24] In contrast, the mean-field theory for the large-n problem has been treated in a more heuristic way, through the use of averaging methods,[19] self-consistency arguments,[19,26] and integral equations.[25] A more detailed review of previous work is given in Section 6.

In this paper we present a rigorous analysis of amplitude death in the mean-field model. Our theorems clarify the mathematical basis for certain results obtained more formally by previous authors.[19,25,26] For example, we give the first proof that the infinite-n system provides an accurate picture of the finite-n system, for sufficiently large n. This work is part of a larger effort in progress, in which other aspects of the dynamics of oscillator arrays will be analyzed.

Model

Following refs. 18, 19, 25, and 26, we study the system of differential equations

$$\frac{dz_j}{dt} = (1 - |z_j|^2 + i\omega_j)\, z_j + K(\langle z \rangle - z_j), \qquad j = 1,..., n \tag{1}$$

where $z_j(t)$ is a complex number which represents the state of the jth oscillator at time t, $K \geqslant 0$ is the coupling strength, $\langle z \rangle$ is the mean of the z_j given by

$$\langle z \rangle = \frac{1}{n} \sum_{i=1}^{n} z_i$$

and the $\omega_j \in R$ are the natural frequencies of the oscillators. The frequencies are assumed to be randomly distributed with a density $g(\omega)$; by going into a rotating frame if necessary, we can assume without loss of generality that g has zero mean.

Equation (1) represents a population of limit cycle oscillators with mean-field coupling; each oscillator is coupled to all the others through the average quantity $\langle z \rangle$. In the absence of coupling, each oscillator has an asymptotically stable limit cycle of radius $|z_j| = 1$ and an unstable equilibrium point at $z_j = 0$. The dynamics of the coupled system is more complex, although one trivial solution is clear from inspection: there is always a static solution with $z_j = 0$ for all j. This solution is the state of "amplitude death" discussed above. Depending on the coupling strength K and the sample $\omega_1,..., \omega_n$, the origin may or may not be stable.

Our goal is determine the precise conditions under which amplitude death is stable. In Section 2, we derive the characteristic equation for (1) at the origin; in Section 3, we discuss the limit of this equation as $n \to \infty$. In the limit, the conditions for stability depend solely on the coupling K and the distribution $g(\omega)$, as shown in Section 4. In Section 5, we consider certain one-parameter families of $g(\omega)$ (depending on the standard deviation σ of g), and explicitly determine the region in (K, σ) space for which amplitude death is stable. The stability boundary is given by a transcendental equation in K and σ. We show that these exact results for the stability boundary are well approximated by a much simpler asymptotic formula. In Section 6 we discuss the relation of our work to previous studies.

2. CHARACTERISTIC EQUATION

The linearization of (1) at $z_j = 0$ is the system

$$\frac{dz_j}{dt} = (1 - K + i\omega_j)\, z_j + K\langle z \rangle, \qquad j = 1,..., n \tag{2}$$

For stability we need all eigenvalues λ_j of (2) to satisfy Re $\lambda_j \leqslant 0$. Let A be the matrix of (2). Then

$$\mathrm{Re}(\mathrm{Tr}\, A) = n(1 - K) + K = n - (n - 1)\, K$$

Hence a necessary condition for stability is

$$\mathrm{Re}(\mathrm{Tr}\, A) \leqslant 0 \Leftrightarrow K \geqslant \frac{n}{n-1}$$

In particular, $K > 1$ is necessary for stability.

Now we derive the characteristic equation for A. To simplify the notation, we set

$$B = A + (K - 1)\, I$$

Let μ denote an eigenvalue of B. Then the eigenvalues of B and A are related by

$$\mu = \lambda + K - 1 \tag{3}$$

The characteristic equation for B is $\det(\mu I - B) = 0$, where the matrix elements of B are given by

$$B_{pq} = \begin{cases} K/n + i\omega_p, & p = q \\ K/n, & p \neq q \end{cases} \tag{4}$$

The following lemma is useful for calculating the characteristic equation of B.

Lemma:

$$\det \begin{pmatrix} 1 + x_1 & x_1 & \cdots & \cdots & x_1 \\ x_2 & 1 + x_2 & \cdots & x_2 & x_2 \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ x_n & x_n & \cdots & x_n & 1 + x_n \end{pmatrix} = 1 + x_1 + x_2 + \cdots + x_n$$

Proof:

$$\det\begin{pmatrix} 1+x_1 & x_1 & \cdots & \cdots & x_1 \\ x_2 & 1+x_2 & \cdots & x_2 & x_2 \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ x_n & x_n & \cdots & x_n & 1+x_n \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ x_2 & 1+x_2 & \cdots & x_2 & x_2 \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ x_n & x_n & \cdots & x_n & 1+x_n \end{pmatrix}$$

$$+ x_1 \det\begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 \\ x_2 & 1+x_2 & \cdots & x_2 & x_2 \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ x_n & x_n & \cdots & x_n & 1+x_n \end{pmatrix}$$

By induction, the first determinant on the right-hand side is $1+x_2+\cdots+x_n$. The contribution of the second determinant is x_1, as can be seen from row reduction. This completes the proof. ▮

We apply this lemma to find the characteristic equation of B.

Proposition:

$$\det(\mu I - B) = (\mu - i\omega_1)\cdots(\mu - i\omega_n)\left[1 - \frac{K}{n}\sum_{j=1}^{n}(\mu - i\omega_j)^{-1}\right] \tag{5}$$

Proof. The elements of B are given by (4). By factoring out a term $\mu - i\omega_j$ from row j, for $j = 1,\ldots, n$, we obtain

$$\det(\mu I - B) = (\mu - i\omega_1)\cdots(\mu - i\omega_n)$$

$$\times \det\begin{pmatrix} 1+x_1 & x_1 & \cdots & \cdots & x_1 \\ x_2 & 1+x_2 & \cdots & x_2 & x_2 \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ x_n & x_n & \cdots & x_n & 1+x_n \end{pmatrix}$$

where

$$x_j = -\frac{K}{n}(\mu - i\omega_j)^{-1}, \qquad j = 1, \ldots, n$$

Applying the lemma now yields the desired result. ▌

3. COMPARISON OF FINITE AND INFINITE SYSTEMS

In general, it is too difficult to find the solutions μ of the characteristic equation (5), because they depend on the particular sample $\omega_1, \ldots, \omega_n$. Fortunately, the problem becomes tractable in the limit $n \to \infty$. In this section, we present a result which ensures that the infinite-n problem provides an accurate picture of the finite-n problem, for n sufficiently large.

The stability type of the origin depends on whether the characteristic equation (5) has roots μ with $\operatorname{Re} \mu \geqslant K - 1 > 0$. Hence we restrict attention to μ in the right half-plane $\operatorname{Re} \mu > 0$. The roots in this domain are given by the solution of $f_n(\mu) = K^{-1}$, where

$$f_n(\mu) = \frac{1}{n} \sum_{j=1}^{n} (\mu - i\omega_j)^{-1} \tag{6}$$

The function $f_n(\mu)$ is analytic in this domain.

Let $n \to \infty$ in (6). Then $f_n(\mu)$ converges, in a sense to be specified below, to the analytic function

$$f(\mu) = \int_{-\infty}^{\infty} (\mu - i\omega)^{-1} g(\omega)\, d\omega \tag{7}$$

Our main theorem states that the location of the roots of $f(\mu) = K^{-1}$ governs the behavior of the finite-n system, with probability $\to 1$ as $n \to \infty$. More precisely:

Theorem 1. (A) Suppose all roots of $f(\mu) = K^{-1}$ satisfy $\operatorname{Re} \mu < K - 1$. Then amplitude death is stable with probability $\to 1$ as $n \to \infty$.

(B) Suppose $f(\mu) = K^{-1}$ has a root μ with $\operatorname{Re} \mu > K - 1$. Then amplitude death is unstable with probability $\to 1$ as $n \to \infty$.

Proof. The theorem follows from Chebyshev's inequality and Rouché's theorem. Using Chebyshev's inequality, we show that $f_n(\mu)$ is close to $f(\mu)$ along a certain line. Then we invoke Rouché's theorem to conclude that the functions $f_n(\mu) - K^{-1}$ and $f(\mu) - K^{-1}$ have the same number of zeros in a certain region, with probability $\to 1$ as $n \to \infty$.

First we claim that $f(\mu)=K^{-1}$ can have only finitely many solutions. If there were an infinite sequence of roots, then this sequence either diverges, or it has a subsequence converging to a limit point. Both cases lead to contradictions: If a subsequence converges to a limit point, then f would have to be a constant function, by analytic continuation; and a sequence of roots cannot possibly go off to infinity, since $f(\mu)\to 0\neq K^{-1}$ as $\mu\to\infty$.

Hence, for all but finitely many $a>0$ the line $\operatorname{Re}\mu=a$ has no roots of $f(\mu)=K^{-1}$ on it. On such a line,

$$\varepsilon=\min_{\operatorname{Re}\mu=a}|f(\mu)-K^{-1}|>0$$

The next step is to bound the difference between $f_n(\mu)$ and $f(\mu)$ on this line. Fix μ. Let

$$X_j=(\mu-i\omega_j)^{-1}$$

for $j=1,\dots,n$. Then $X_1,\dots,X_n$ are independent and identically distributed, with

$$\mathrm{E}X_j=f(\mu)$$

and

$$\frac{1}{n}\sum_{j=1}^{n}X_j=f_n(\mu)$$

Set

$$\sigma_\mu^2=\operatorname{Var}X_j=\int_{-\infty}^{\infty}|(\mu-i\omega)^{-1}-f(\mu)|^2\,g(\omega)\,d\omega$$

Then Chebyshev's inequality implies

$$\operatorname{Prob}(|f_n(\mu)-f(\mu)|<\varepsilon)>1-\frac{\sigma_\mu^2}{n\varepsilon^2}$$

Since σ_μ^2 is uniformly bounded on the domain $\operatorname{Re}\mu\geqslant a$, we conclude that

$$\lim_{n\to\infty}\operatorname{Prob}(|f_n(\mu)-f(\mu)|<\varepsilon\ \forall\mu\text{ s.t. }\operatorname{Re}\mu=a)=1$$

Hence, with probability $\to 1$,

$$|f_n(\mu)-f(\mu)|<|f_n(\mu)-K^{-1}|\quad\forall\mu\text{ s.t. }\operatorname{Re}\mu=a$$

Now the functions $f_n(\mu)$ and $f(\mu)$ are analytic at infinity as well. Hence we may apply Rouché's theorem to conclude that the equations $f_n(\mu) = K^{-1}$ and $f(\mu) = K^{-1}$ have the same number of roots (counting multiplicities) in the region $\operatorname{Re} \mu > a$, with probability $\to 1$ as $n \to \infty$.

To prove (A), choose the line $\operatorname{Re} \mu = a$ to the left of $\operatorname{Re} \mu = K - 1$, but to the right of any roots of $f(\mu) = K^{-1}$.

To prove (B), choose the line $\operatorname{Re} \mu = a$ to the right of $\operatorname{Re} \mu = K - 1$, but to the left of some root of $f(\mu) = K^{-1}$ with $\operatorname{Re} \mu > K - 1$. ∎

Location of the Eigenvalues

As a byproduct of the argument above, we can show that all but a bounded number of the eigenvalues λ will lie close to the vertical line $\operatorname{Re} \lambda = 1 - K$. For any region $\operatorname{Re} \mu > a > 0$, the equation $f_n(\mu) = K^{-1}$ will have (with probability $\to 1$ as $n \to \infty$) a *bounded* number of solutions, and the bound is *independent of n*. Moreover, by considering the real and imaginary parts of the equation $f_n(\mu) = K^{-1}$, one can see that if μ is an eigenvalue of B, then

$$\operatorname{Re}(\mu) \geqslant 0 \qquad \text{and} \qquad \omega_{\min} \leqslant \operatorname{Im}(\mu) \leqslant \omega_{\max}$$

where $\omega_{\min}$ and $\omega_{\max}$ are the least and greatest ω's in the sample $\omega_1, \ldots, \omega_n$. Hence all but a bounded number of eigenvalues μ will be very close to the support of $g(\omega)$ on the imaginary axis. The result for λ follows from (3).

We now illustrate these results with a numerical example. Figure 1 plots the eigenvalues $\lambda_1, \ldots, \lambda_n$ for the system (2) for the case of $n = 25$ oscillators with coupling strength $K = 2$ and random frequencies $\omega_1, \ldots, \omega_n$ sampled from a uniform distribution on the interval $[-\gamma, \gamma]$. Figure 1a shows that when $\gamma = 2$, one of the eigenvalues has positive real part. Hence amplitude death would be unstable for this choice of parameters. Figure 1b shows that when the width is increased to $\gamma = 2.4$, all the eigenvalues have negative real parts and amplitude death becomes stable. In both figures, all of the other eigenvalues lie close to the vertical line $\operatorname{Re} \lambda = 1 - K$ $(= -1)$ and satisfy $|\operatorname{Im} \lambda| \leqslant \gamma$, as expected from the results above.

Infinite-Dimensional Analog

The infinite-dimensional analog of the matrix B helps to explain the location of the eigenvalues. In this case, the limiting operator has a *continuous spectrum* given by the support of $g(\omega)$ on the imaginary axis, as we shall show now.

Let h be any function in L^1 relative to the measure $g(\omega)\,d\omega$, and consider the linear operator B on L^1 defined by

$$(Bh)(x) = ixh(x) + K\int_{-\infty}^{\infty} h(\omega)\, g(\omega)\, d\omega$$

Then

$$(\mu I - B)\, h(x) = (\mu - ix)\, h(x) - K\int_{-\infty}^{\infty} h(\omega)\, g(\omega)\, d\omega$$

Therefore $h(x)$ is in the kernel of $\mu I - B$ if and only if

$$h(x) = K(\mu - ix)^{-1}\int_{-\infty}^{\infty} h(\omega)\, g(\omega)\, d\omega \tag{8a}$$

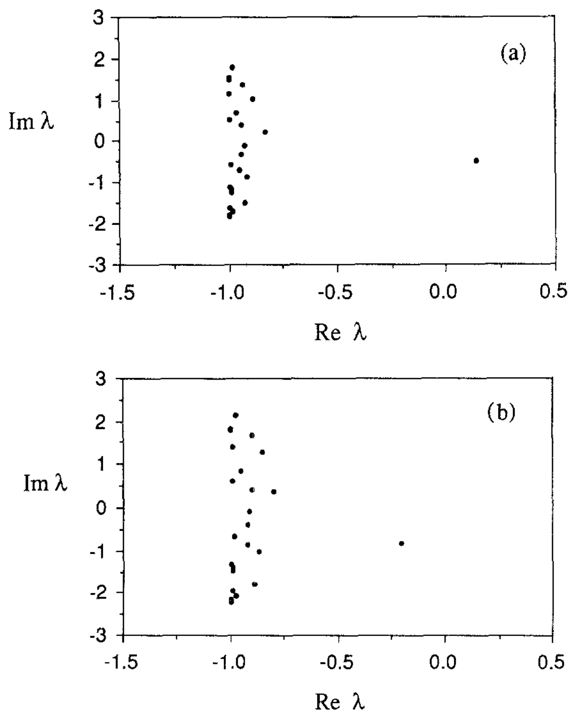

Fig. 1. Eigenvalues of the system (2), for $n = 25$ oscillators, coupling strength $K = 2$, and random frequencies sampled from a uniform distribution on $[-\gamma, \gamma]$. (a) $\gamma = 2.0$; note that one of the eigenvalues has positive real part. (b) $\gamma = 2.4$; all eigenvalues have negative real part, so amplitude death is stable. In both (a) and (b), all but one of the eigenvalues lie close to the vertical line $\operatorname{Re} \lambda = 1 - K$. This line contains the continuous spectrum of the limiting operator obtained in the infinite-n limit.

and

$$K^{-1} = \int_{-\infty}^{\infty} (\mu - i\omega)^{-1} \, g(\omega) \, d\omega \tag{8b}$$

From (7) and (8b) we conclude that $f(\mu) = K^{-1}$ is the equation for the *discrete* spectrum of B. Note that (8b) is essentially a self-consistency equation, a familiar concept in mean-field models.[14,19,26,28,29]

But $\mu I - B$ may fail to be surjective, even if its kernel is $\{0\}$. Solving

$$(\mu I - B) \, h(x) = q(x)$$

gives

$$h(x) = (\mu - ix)^{-1} \, q(x) + C(\mu - ix)^{-1}$$

where C is some constant. This is possible for *any* choice of $q(x)$ exactly when $\mu \neq i\omega$, for ω in the support of $g(\omega)$. Hence the *continuous* spectrum of B is $\{i\omega\colon g(\omega) \neq 0\}$.

The upshot is that for $n \to \infty$, the majority of the eigenvalues are close to the continuous spectrum of the limiting operator.

4. CRITICAL CONDITION FOR STABILITY

In this section, we refine conditions (A) and (B) in Theorem 1.

Theorem 2.
Assume that the density $g(\omega)$ is an even function which is nonincreasing on $[0, \infty)$. Then:

(A) Amplitude death is stable with probability $\to 1$ as $n \to \infty \Leftrightarrow f(K-1) < K^{-1}$.

(B) Amplitude death is unstable with probability $\to 1$ as $n \to \infty \Leftrightarrow f(K-1) > K^{-1}$.

Remarks. Theorem 2 allows us to determine whether amplitude death is stable by calculating one equation. Moreover, we now see that *the critical condition for stability in the large-n limit is*

$$f(K-1) = K^{-1} \tag{9}$$

Conditions equivalent to (9) have also been derived by Shiino and Frankowicz[26] and Ermentrout.[25]

Proof. We claim that it suffices to prove two facts:

(i) The equation $f(\mu) = K^{-1}$ has only *real* solutions.

(ii) $f(\mu)$ is strictly decreasing for $\mu > 0$.

To see that these two facts imply the desired result, suppose first that $f(K-1) < K^{-1}$. Then either there is no solution to $f(\mu) = K^{-1}$ (Fig. 2a), in which case the discrete spectrum is empty, or there is a unique solution and it satisfies $\mu < K-1$ (Fig. 2b). In either case, Theorem 1(A) implies that amplitude death is stable with probability $\to 1$ as $n \to \infty$. Now suppose that $f(K-1) > K^{-1}$. Then, since $f(\mu) \to 0$ as $\mu \to \infty$, there exists a positive solution of $f(\mu) = K^{-1}$, and it satisfies $\mu > K-1$ (Fig. 2c). Hence, by Theorem 1(B), amplitude death is unstable with probability $\to 1$ as $n \to \infty$. Thus, (i) and (ii) imply the desired result, as claimed.

To prove (i), write $\mu = \alpha + i\beta$. Then (7) implies

$$\operatorname{Im} f(\mu) = \int_{-\infty}^{\infty} \frac{\omega - \beta}{\alpha^2 + (\beta - \omega)^2}\, g(\omega)\, d\omega$$

Since $f(\bar{\mu}) = \overline{f(\mu)}$, we can assume without loss of generality that $\beta \geqslant 0$. Then

$$\int_{-\infty}^{\infty} \frac{\omega - \beta}{\alpha^2 + (\beta - \omega)^2}\, g(\omega)\, d\omega = \int_{-\infty}^{\infty} \frac{\omega}{\alpha^2 + \omega^2}\, g(\omega + \beta)\, d\omega$$

$$= \int_{0}^{\infty} \frac{\omega}{\alpha^2 + \omega^2}\, [g(\omega + \beta) - g(\omega - \beta)]\, d\omega$$

Now we break the range of integration into two parts, as follows:

$$\operatorname{Im} f(\mu) = \int_{0}^{\beta} \frac{\omega}{\alpha^2 + \omega^2}\, [g(\omega + \beta) - g(\omega - \beta)]\, d\omega$$

$$+ \int_{\beta}^{\infty} \frac{\omega}{\alpha^2 + \omega^2}\, [g(\omega + \beta) - g(\omega - \beta)]\, d\omega$$

In the first integral, $0 \leqslant \beta - \omega \leqslant \beta + \omega$, and in the second integral, $0 \leqslant \omega - \beta \leqslant \omega + \beta$. Hence both integrands are nonpositive. In fact the second integral is strictly negative unless $\beta = 0$. [Otherwise we would need $g(\omega - \beta) = g(\omega + \beta)$ for almost all $\omega \geqslant \beta$, but this is impossible, since $g(\omega) \to 0$ as $\omega \to \infty$.] Hence, if $\operatorname{Im} f(\mu) = 0$, we must have $\beta = 0$. In particular, all solutions of $f(\mu) = K^{-1}$ are real.

To prove (ii), suppose $0 < \mu < \nu$. Write $\nu = r\mu$ with $r > 1$. Then (7) yields

$$f(\nu) = f(r\mu) = \int_{-\infty}^{\infty} \frac{r\mu}{r^2\mu^2 + \omega^2}\, g(\omega)\, d\omega$$

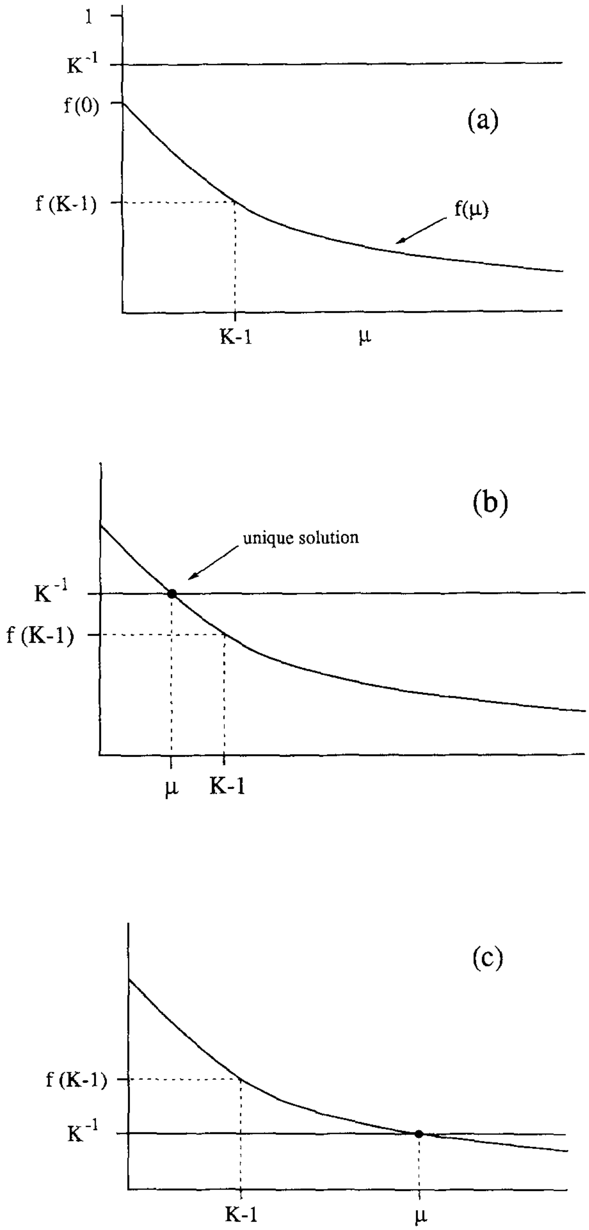

Fig. 2. (a) When $f(0) < K^{-1}$, there is no solution to $f(\mu) = K^{-1}$. (b) When $f(K-1) < K^{-1} < f(0)$, there exists a unique solution to $f(\mu) = K^{-1}$, and it satisfies $\mu < K-1$. (c) When $f(K-1) > K^{-1}$, there exists a unique solution to $f(\mu) = K^{-1}$, and it satisfies $\mu > K-1$.

Write $\omega = r\rho$. Then $d\omega = r\,d\rho$ and

$$f(v) = \int_{-\infty}^{\infty} \frac{\mu}{\mu^2 + \rho^2}\, g(r\rho)\, d\rho$$

This expression is the same as that for $f(\mu)$, but with $g(r\rho)$ in place of $g(\rho)$. Since $g(r\rho) \leqslant g(\rho)$ for all ρ, and equality almost everywhere is impossible since $g(\rho) \to 0$ as $\rho \to \infty$, we conclude that $f(v) < f(\mu)$. ▌

5. ASYMPTOTICS AND EXAMPLES

Theorem 2 shows that in the infinite-n limit, amplitude death is stable in a region of parameter space defined by $K > 1$ and $f(K-1) < K^{-1}$. What does this stability region look like? In this section, we derive an asymptotic expansion for the boundary curve (9), under the assumptions that $K \gg 1$ and that $g(\omega)$ has finite moments of all orders. We then compare the asymptotic expansion to exact results obtained in two cases where we can calculate the boundary curve (9) explicitly. A two-term asymptotic expansion is found to work remarkably well, even when $K = O(1)$.

5.1. Asymptotic Behavior of Stability Boundary

Proposition. Suppose that the density $g(\omega)$ is an even function with finite moments of all orders. Let σ and m_4 be defined as follows:

$$\sigma^2 = \int_{-\infty}^{\infty} \omega^2 g(\omega)\, d\omega < \infty$$

$$m_4 = \frac{1}{\sigma^4} \int_{-\infty}^{\infty} \omega^4 g(\omega)\, d\omega < \infty$$

Then the solution of (9) satisfies

$$\sigma \sim K^{1/2} + (m_4/2 - 1)\, K^{-1/2} + O(K^{-3/2}) \tag{10}$$

as $K \to \infty$.

Proof. We develop a formal power series solution of (9) by expanding $f(K-1)$ in even powers of $\omega/(K-1)$, as follows:

$$\begin{aligned} f(K-1) &= \int_{-\infty}^{\infty} \frac{(K-1)\, g(\omega)\, d\omega}{(K-1)^2 + \omega^2} \\ &= \frac{1}{K-1} \int_{-\infty}^{\infty} g(\omega) \sum_{j=0}^{\infty} (-1)^j \left(\frac{\omega}{K-1}\right)^{2j} d\omega \end{aligned}$$

Then (9) can be rearranged to yield

$$\frac{(K-1)^2}{K}=\sigma^2-\frac{\sigma^4 m_4}{(K-1)^2}+O\left(\frac{\sigma^6}{(K-1)^4}\right)$$

Solving this equation for σ gives the desired result (10). ▮

5.2. Corner of the Stability Region

We now derive an expression for the "corner" of the stability region. That is, we find the limiting value of σ as $K \to 1$ along the boundary curve (9). This result complements the asymptotic formula (10), which holds for large K.

Let G be a scaled version of g with standard deviation 1. That is,

$$g(\omega)=(1/\sigma)\, G(\omega/\sigma)$$

We claim that

$$\sigma \to \pi G(0) \qquad \text{as} \quad K \to 1^+ \tag{11}$$

along the boundary curve defined by $f(K-1)=K^{-1}$. To see this, let $\mu=K-1$ and consider the limit of $f(\mu)$ as $\mu \to 0^+$. Because G is even, we have

$$f(\mu)=\frac{1}{\sigma}\int_{-\infty}^{\infty}\frac{\mu}{\mu^2+\omega^2}\, G\left(\frac{\omega}{\sigma}\right) d\omega$$

The point is that as $\mu \to 0^+$, the kernel $\mu/(\mu^2+\omega^2)$ approaches a "delta function" with respect to ω: its integral over the real line is π, for all $\mu>0$, and it becomes increasingly sharply peaked near $\omega=0$ as $\mu \to 0$ from above. So $f(\mu) \to \pi G(0)/\sigma$ as $\mu \to 0^+$. Hence (9) implies that $\pi G(0)/\sigma \to 1$ as $K \to 1^+$, which yields the desired result (11).

Example 1. Uniform Distribution. Consider the uniform distribution with density given by

$$g(\omega)=\begin{cases} 1/(2\gamma), & |\omega| \leqslant \gamma \\ 0, & |\omega|>\gamma \end{cases}$$

Then (7) can be integrated and the critical condition (9) becomes

$$\gamma \cot(\gamma/K)+1-K=0 \tag{12}$$

Here $\gamma=\sqrt{3}\,\sigma$, where σ is the standard deviation as above. Equation (12) can be solved numerically to obtain γ and hence σ as a function of K.

Figure 3 plots the region in the (K, σ) plane where amplitude death is stable. Remarkably, the approximate solution (10) yields values of σ that are within 1% of the exact solution (12) for all $K \geqslant 1$. The greatest error occurs at the corner of the stability region: for $K = 1$ the exact value from (11) is $\sigma = \pi/(2\sqrt{3}) \approx 0.907$, and the value predicted by (10) is $\sigma \approx K^{1/2} - 0.1K^{-1/2} = 0.900$. (Here we have used the fact that $m_4 = 9/5$ for the uniform distribution.)

Example 2. Triangle Distribution. Our second example is the triangle distribution with density given by

$$g(\omega) = \begin{cases} \dfrac{\gamma - |\omega|}{\gamma^2}, & |\omega| \leqslant \gamma \\ 0, & |\omega| \geqslant \gamma \end{cases}$$

Again (7) can be integrated and the condition (9) now becomes

$$\frac{1}{K} = \frac{2}{\gamma} \tan^{-1}\left(\frac{\gamma}{K-1}\right) - \frac{K-1}{\gamma^2} \ln\left[1 + \left(\frac{\gamma}{K-1}\right)^2\right] \tag{13}$$

The triangle distribution has $\gamma = \sqrt{6}\,\sigma$ and $m_4 = 12/5$. Hence the asymptotic approximation (10) becomes

$$\sigma \approx K^{1/2} + 0.2K^{-1/2}$$

Figure 4 shows that the asymptotic result is again very close to the exact result.

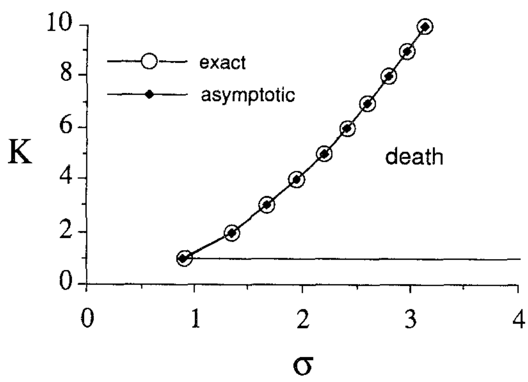

Fig. 3. Stability region for amplitude death, for the case of a uniform distribution $g(\omega)$ with standard deviation σ. The region is bounded by the line $K = 1$ and the curve (12). The numerical solution of (12) is well approximated by the large-K asymptotic solution (10), even when $K = 1$.

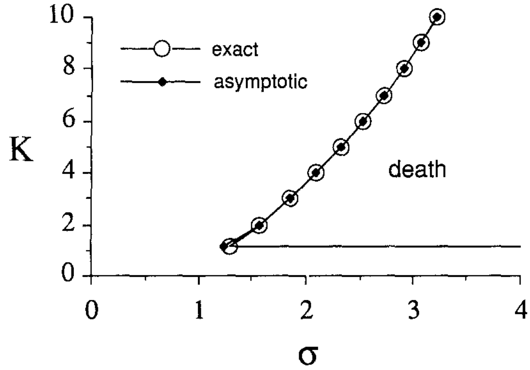

Fig. 4. Stability region for amplitude death, for the case of a triangle distribution $g(\omega)$ with standard deviation σ. The region is bounded by the line $K = 1$ and the curve (13). The numerical solution of (13) is closely approximated by the asymptotic solution (10).

6. RELATION TO PREVIOUS WORK

As mentioned in the Introduction, a number of previous authors have discussed the phenomenon of amplitude death. The work of Yamaguchi and Shimizu,[19] Ermentrout,[25] and Shiino and Frankowicz[26] is particularly relevant, because they studied Eq. (1) or equations equivalent to it.

Yamaguchi and Shimizu[19] studied a system of weakly nonlinear van der Pol oscillators with mean field coupling, random intrinsic frequencies, and external white noise. Using the method of averaging and the slaving principle of Haken,[31] they reduced their system to (1), with an additional noise term. This application of the slaving principle was argued to be valid only if $K > 1$ (in our notation); as we have shown, this is one of the necessary conditions for stable amplitude death (which Yamaguchi and Shimizu called "mechanical bifurcation"). To study self-synchronization of the oscillators as well as the stability of amplitude death, they derived an approximate evolution equation for the growth of the order parameter $\langle z \rangle$. This evolution equation allowed them to calculate the stability boundary for amplitude death for various distributions $g(\omega)$, including some of those later discussed by Shiino and Frankowicz[26] and Ermentrout.[25]

Shiino and Frankowicz[26] studied the dynamics of the system (1) for both $K < 1$ and $K > 1$ for Lorentzian and Gaussian distributions $g(\omega)$. They found the stability region for amplitude death (which they called the "quenched" state) by a self-consistent mean-field argument: they sought phase-locked solutions of (1) with some fixed value of $\langle z \rangle$. In a rotating frame, the problem reduces to a static situation where the locked position of each oscillator depends on $\langle z \rangle$, but also determines $\langle z \rangle$. A branch of self-consistent solutions with $\langle z \rangle \neq 0$ is born along the stability boundary of amplitude death. This method gives the boundary curve (9), but provides no information about stability, which Shiino and Frankowicz[26] determined numerically.

Ermentrout[25] used the methods of classical applied mathematics, rather than statistical mechanics, to analyze (1). He showed that as $n \to \infty$, the long-time behavior of $\langle z \rangle$ is governed by a certain integral equation. This equation has exponentially decaying solutions when two conditions are satisfied: one is $K > 1$ and the other is a condition on the Fourier transform of g, which is equivalent to our condition $f(K-1) < K^{-1}$. He calculated the stability boundaries for a number of specific distributions, and gave asymptotic results when the frequencies were not random, but evenly spaced on some interval. One of Ermentrout's most important findings[25] is that amplitude death is a robust phenomenon—his numerical results indicate that amplitude death occurs in a wide variety of systems, and does not depend on the special symmetries or infinite-range coupling in (1).

The novel aspect of our work is that we determine the stability of amplitude death by calculating eigenvalues. Our techniques are both elementary and rigorous, but they had not been applied to this problem before. The results presented here clarify a number of points that may have seemed puzzling in the earlier work. For example, one would expect that n inequalities would be needed to ensure the stability of the origin in the n-dimensional dynamical system (1), yet previous authors have claimed that only two inequalities are needed: essentially $K > 1$ and $f(K-1) < K^{-1}$. Our Theorem 2 allows one to understand why two inequalities are enough—for large n, the finite-n system is well approximated by the infinite-n system, and only two conditions are needed to ensure that death is stable for the infinite-n system. The condition $f(K-1) < K^{-1}$ keeps the discrete spectrum from causing instability and the condition $K > 1$ similarly controls the continuous spectrum.

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